

Euclidean Pseudoduality and Boundary Conditions in Sigma Models

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Abstract

We discuss pseudoduality transformations in two dimensional conformally invariant classical sigma models, and extend our analysis to a given boundaries of world-sheet, which gives rise to an appropriate framework for the discussion of the pseudoduality between D-branes. We perform analysis using the Euclidean spacetime and show that structures on the target space can be transformed into pseudodual manifold identically. This map requires that torsions and curvatures related to individual spaces are the same when connections are riemannian. Boundary pseudoduality imposes locality condition.

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1 Introduction

The term ‘duality’ is widely used in physics literature to express that two different systems turn out to be equivalent when there is a duality transformation between these systems. In string theory people use the term ‘target space duality’ [1, 2, 3, 4] if there is a canonical transformation between target spaces in which strings move. This transformation preserves the hamiltonian.

In recent years a new type of duality transformation called pseudoduality was suggested by Curtright and Zachos, Ivanov and Alvarez¹ [5, 6, 7, 8]. This new topical issue is quite interesting since it addresses duality transformation on the world-sheet as distinct from the usual duality transformation on the target space. The prominent feature of pseudoduality is to preserve the

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¹The term ‘pseudoduality’ was first introduced by Curtright and Zachos in [7], used by Ivanov [8], and developed by Alvarez [5, 6]

stress-energy tensor, and therefore in principle not a canonical transformation [5, 6]. This ‘on-shell’ duality transformation is carried out by mappings between the solutions of the equations of motion.

In our prior research we analysed pseudoduality in symmetric space sigma models [16] based on Lie group valued fields, and extended it to supersymmetric case in [9, 10, 11]. In these papers, there were some global problems traced back to the signature of the worldsheet, especially in supersymmetrized worldsheet. In order to designate and solve this concern, we work with Wick rotated worldsheet in the present paper.

Recent studies [5, 9, 10, 11] about pseudoduality in sigma models revealed that constructing pseudoduality in the worldsheet with lorentzian signature is not a pleasant approach since the negative sign in the pseudoduality expressions arising from lorentzian point of view leads to the vanishing torsion in both manifolds and more importantly when supersymmetry is imposed, to non invertible mapping which maps all the points on one manifold to only one point on the pseudodual manifold. This is the point that has vanishing Riemann connection. It is understood [9] that it is better to perform the pseudoduality between worldsheets which have general (non-Lorentzian) signatures. To realize this goal, in this paper we set up a simplified version, the Euclidean pseudoduality transformations. Accordingly we extend our analysis to the given boundaries of the world-sheet coordinates in the classical sigma models.

It was observed [9] that expressing pseudoduality in standard lightcone coordinates causes the geometry of target spaces to be torsion free, and sigma model is not globally defined on the pseudodual manifold. Therefore it is not invertible. To fix this problem we will introduce alternative pseudoduality expressions in the Euclidean worldsheet which is parameterized by τ and σ . We point out that alternative pseudodualities adjust the curvatures of target spaces by means of a modified connection so that target spaces are globally well-defined and constructed as diffeomorphic and dual symmetric spaces respectively with respect to the modified connections in these cases, which are not just a characteristic of sigma models. Pseudoduality also imposes that torsions of the target spaces on which sigma models are based produce infinitely many conditions related to their covariant derivatives in the first case, and vanish in case of symmetric spaces which coincides with the results obtained in literature.

2 The Framework

The sigma model with target space M , metric g and antisymmetric 2-form b is denoted by (M, g, b) and has the action in the Euclidean worldsheet Σ [12]

$$\begin{aligned}
S &= \frac{1}{2} \int_{\Sigma} d^2\sigma \sqrt{h} [h^{\mu\nu} \partial_{\mu} x^i \partial_{\nu} x^j g_{ij}(x) + i\epsilon^{\mu\nu} \partial_{\mu} x^i \partial_{\nu} x^j b_{ij}(x)] \\
&= \int_{\Sigma} d^2\sigma \left[\left(\frac{1}{2} g_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} + \frac{1}{2} g_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \sigma} \right) + i b_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma} \right]
\end{aligned} \tag{1}$$

where $x : \Sigma \longrightarrow M$ is specified locally by the functions $x^i(\sigma)$ giving the dependence of coordinates x^i of M on the coordinates σ^{μ} of Σ . The worldsheet Σ is endowed with the Euclidean metric $h_{\mu\nu}$ with $h = |\det(h_{\mu\nu})|$. The globally defined closed 3-form H is locally given by $H = db$. Notice that Euclidean version of the action is obtained by Wick rotation of the Lorentzian case (see appendix) in the case in which h is flat. Since $g_{ij}(x)$ and $b_{ij}(x)$ have real components, the term involving b is pure imaginary so that action is complex.

We will assume that sigma model is defined on a region U of Σ with boundary ∂U . Equations of motion following from this action in the bulk space will be

$$x_{\tau\tau}^i + x_{\sigma\sigma}^i = -\Gamma_{jk}^i (x_{\tau}^j x_{\tau}^k + x_{\sigma}^j x_{\sigma}^k) + i H_{jk}^i x_{\tau}^j x_{\sigma}^k \tag{2}$$

with the corresponding Dirichlet and Neumann boundary conditions respectively

$$\delta x^i = 0 \tag{3}$$

$$x_{\sigma}^i - i b_j^i x_{\tau}^j = 0 \tag{4}$$

where we defined $x_{\tau} \equiv \frac{\partial x}{\partial \tau}$ and $x_{\sigma} \equiv \frac{\partial x}{\partial \sigma}$. We would like to relate the sigma model (M, g, b) to a different one $(\tilde{M}, \tilde{g}, \tilde{b})$ by means of these equations. The pseudodual model will be represented by $(\tilde{M}, \tilde{g}, \tilde{b})$ and similar expressions may be written on $(\tilde{M}, \tilde{g}, \tilde{b})$ using tilde. As it is well known pseudoduality equations are best formulated on the orthonormal coframe bundle ².

We choose an orthonormal frame $\{\theta^i\}$ with the riemannian connection ω_j^i defined on the worldsheet as

$$\theta^i = x_a^i d\pi^a = x_{\tau}^i d\tau + x_{\sigma}^i d\sigma \tag{5}$$

where the worldsheet coordinates are given by $\pi = (\tau, \sigma)$. The indices in the middle of alphabet denote coordinates on the target manifold while indices a, b, c, \dots represent coordinates of the worldsheet. In what follows we will construct two different pseudoduality equations. The first one yields the coordinate diffeomorphisms and the second case restricts manifolds to symmetric spaces.

²Orthonormal coframe bundle is defined on $SO(M) = M \times SO(n)$.

2.1 Case I: Pseudoduality to Coordinate Diffeomorphisms

In this case we assume that pseudoduality equations are defined on wick rotated world-sheet using the pullback bundle of target space as $\tilde{\theta} = T\theta$. These are explicitly written as

$$\tilde{x}_\tau^i = T_j^i x_\tau^j \quad (6)$$

$$\tilde{x}_\sigma^i = T_j^i x_\sigma^j \quad (7)$$

in order to better understand these equations we will inquire the integrability conditions of these equations as in [5, 9]. We define the covariant derivative of x_a^i

$$dx_a^i + \omega_j^i x_a^j = x_{ab}^i d\pi^b \quad (8)$$

Cartan structural equations are given by

$$d\theta^i + \omega_j^i \wedge \theta^j = 0 \quad (9)$$

$$d\omega_j^i + \omega_k^i \wedge \omega_j^k = \Omega_j^i \quad (10)$$

where $\Omega_j^i = \frac{1}{2} R_{jkl}^i \theta^k \wedge \theta^l$ is the curvature two-form. We take the exterior derivative of (6) and (7) and use the covariant derivative (8) to obtain the following

$$\tilde{x}_{\tau b}^i d\pi^b = [dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\tau^k + T_j^i x_{\tau b}^j d\pi^b \quad (11)$$

$$\tilde{x}_{\sigma b}^i d\pi^b = [dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\sigma^k + T_j^i x_{\sigma b}^j d\pi^b \quad (12)$$

These two equations are intriguing and lead to take advantage of equations of motion and the desired integrability conditions for the pseudoduality equations. As opposed to the method followed in [9, 10] we first and foremost wedge the first equation (11) by $d\tau$ and the second (12) by $d\sigma$, and use $x_{\tau\sigma} = x_{\sigma\tau}$ (similarly $\tilde{x}_{\tau\sigma} = \tilde{x}_{\sigma\tau}$) to get

$$[dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\tau^k \wedge d\tau + [dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\sigma^k \wedge d\sigma = 0 \quad (13)$$

For the sake of clarity we split the core part of this equation as $[dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] = \mathcal{A}_{k\tau}^i d\tau + \mathcal{B}_{k\sigma}^i d\sigma$. Therefore, with the use of (5), (13) can be rewritten as

$$\mathcal{C}_{ka}^i \theta^k \wedge d\pi^a := \mathcal{A}_{k\tau}^i \theta^k \wedge d\tau + \mathcal{B}_{k\sigma}^i \theta^k \wedge d\sigma = 0$$

We consider first the “weak” case where $\mathcal{C}_{ka}^i = 0$, which requires $\mathcal{A}_{k\tau}^i = 0$ and $\mathcal{B}_{k\sigma}^i = 0$. Therefore we come up with the first integrability condition

$$[dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] = 0 \quad (14)$$

Notice that we have not still made use of the equations of motion for sigma models. This leads to the conclusion that (14) is not special to just sigma models but a property of pseudoduality

itself. What characterizes pseudoduality in sigma models is obtained by wedging (11) by $d\sigma$ and (12) by $d\tau$, and subtracting the resulting equations to get

$$(\tilde{x}_{\tau\tau}^i + \tilde{x}_{\sigma\sigma}^i)d\tau \wedge d\sigma = T_j^i(x_{\tau\tau}^j + x_{\sigma\sigma}^j)d\tau \wedge d\sigma \quad (15)$$

$d\tau \wedge d\sigma$ is trivial and can be cancelled out. Inserting the equations of motion (2) one obtains

$$-\tilde{\Gamma}_{jk}^i(\tilde{x}_\tau^j \tilde{x}_\tau^k + \tilde{x}_\sigma^j \tilde{x}_\sigma^k) + i\tilde{H}_{jk}^i \tilde{x}_\tau^j \tilde{x}_\sigma^k = -T_j^i \Gamma_{mn}^j (x_\tau^m x_\tau^n + x_\sigma^m x_\sigma^n) + iT_j^i H_{mn}^j x_\tau^m x_\sigma^n \quad (16)$$

Notice that this equation consists of symmetric and antisymmetric parts, which leads to the decomposition into two distinct equations. Using pseudoduality equations (6) and (7) in these resulting expressions yields the remaining integrability conditions which are special to sigma models

$$T_j^i \Gamma_{mn}^j = \tilde{\Gamma}_{jk}^i T_m^j T_n^k \quad (17)$$

$$T_j^i H_{mn}^j = \tilde{H}_{jk}^i T_m^j T_n^k \quad (18)$$

These equations can be investigated further by taking exterior derivatives. Exterior derivative of (17) together with condition (14) yields that

$$T_j^i \Omega_n^j = \tilde{\Omega}_n^i T_n^j \quad (19)$$

where we defined $\omega_j^i := \Gamma_{kj}^i \theta^k$, $\Gamma_j^i := \Gamma_{kj}^i \theta^k$ and used the curvature two form $d\Omega_j^i = d\tilde{\Omega}_j^i = d\Gamma_j^i + \Gamma_k^i \wedge \Gamma_j^k$ (similarly for $\tilde{\omega}_j^i := \tilde{\Gamma}_{kj}^i \tilde{\theta}^k$, $\tilde{\Gamma}_j^i := \tilde{\Gamma}_{kj}^i \tilde{\theta}^k$ and $d\tilde{\Omega}_j^i = d\tilde{\Gamma}_j^i + \tilde{\Gamma}_k^i \wedge \tilde{\Gamma}_j^k$). This requires that curvatures of pseudodual manifolds are related to each other by

$$T_j^i R_{k\ell m}^j = \tilde{R}_{j\ell p}^i T_k^j T_m^\ell T_n^p \quad (20)$$

One may continue taking additional exterior derivatives to understand the integrability conditions. But notice that (17) causes (14) to reduce to the form $dT = 0$, which yields that T is constant. Hence it is understood that T is just a constant change of bulk coordinates, $x^i = T_j^i x^j$, which gives an obvious interpretation of the equations (17), (18) and (20). A constant change of coordinates is an obvious pseudoduality, since the sigma-model is invariant under bulk diffeomorphisms. Therefore, torsions and curvatures together with their covariant derivatives are equivalent to each other as expected, i.e.

$$\begin{aligned} R_{jkl}^i &= \tilde{R}_{jkl}^i, & DR_{jkl}^i &= \tilde{D}\tilde{R}_{jkl}^i, & D \dots DR_{jkl}^i &= \tilde{D} \dots \tilde{D}\tilde{R}_{jkl}^i, \\ H_{jk}^i &= \tilde{H}_{jk}^i, & DH_{jk}^i &= \tilde{D}\tilde{H}_{jk}^i, & D \dots DH_{jk}^i &= \tilde{D} \dots \tilde{D}\tilde{H}_{jk}^i, \end{aligned} \quad (21)$$

where $D = \tilde{D}$ is the covariant derivative with respect to the associated riemannian connection.

If one uses the general case that the components of \mathcal{C}_{ka}^i are related to each other by the relation (13), and follow the same steps as above one obtains the following conditions

$$T_j^i H_{mn}^j = \tilde{H}_{jk}^i T_m^j T_n^k \quad (22)$$

$$dT_j^i + \tilde{\omega}_k^i T_j^k - T_k^i \omega_j^k + T_k^i \Gamma_{lj}^k \theta^l - \tilde{\Gamma}_{kl}^i T_m^k T_j^l \theta^m = 0 \quad (23)$$

The first result is the same as (18) and reveals the relation between torsions. The main difference with the “weak” case comes with the second relation and in order to better understand it we define a new (modified) connection $\xi_j^i := \omega_j^i - \Gamma_{kj}^i \theta^k$ (and $\tilde{\xi}_j^i := \tilde{\omega}_j^i - \tilde{\Gamma}_{kj}^i \tilde{\theta}^k$), which leads (23) to

$$dT_j^i + \tilde{\xi}_k^i T_j^k - T_k^i \xi_j^k = 0 \quad (24)$$

Hence it is manifest that the “weak” case corresponds to $\xi = \tilde{\xi} = 0$. Since the characteristics of the pseudoduality is encoded in the transformation map T , it is required to further seek out the integrability of (24), which produces that

$$T_j^i (\Omega_\xi)_k^j = (\tilde{\Omega}_\xi)_j^i T_k^j \quad (25)$$

where we defined a new (modified) curvature two form³ $(\Omega_\xi)_j^i := \Omega_j^i - \Omega_j^{\prime i} := \frac{1}{2}(R_\xi)_{jkl}^i \theta^k \wedge \theta^l$ (and the same relation with tilde on pseudodual manifold). Therefore, the relation between curvatures is similar to (20) and given by

$$T_j^i (R_\xi)_{kmn}^j = (\tilde{R}_\xi)_{j\ell p}^i T_k^j T_m^\ell T_n^p \quad (26)$$

One can work out further integrability of (26) using (24) to obtain that

$$T_j^i (D_\xi R_\xi)_{k\ell m}^j = (\tilde{D}_{\tilde{\xi}} \tilde{R}_\xi)_{j\ell p}^i T_k^j T_m^\ell T_n^p, \quad T_j^i (D_{\tilde{\xi}} \dots D_\xi R_\xi)_{k\ell m}^j = (\tilde{D}_{\tilde{\xi}} \dots \tilde{D}_{\tilde{\xi}} \tilde{R}_\xi)_{j\ell p}^i T_k^j T_m^\ell T_n^p \quad (27)$$

where D_ξ ($\tilde{D}_{\tilde{\xi}}$) is the covariant derivatives with respect to the modified connection ξ ($\tilde{\xi}$), and defined by

$$(D_\xi R_\xi)_{jkl}^i := d(R_\xi)_{jkl}^i + (R_\xi)_{jkl}^q \xi_q^i - (R_\xi)_{qkl}^i \xi_j^q - (R_\xi)_{jqk}^i \xi_\ell^q - (R_\xi)_{jkq}^i \xi_\ell^q$$

Therefore, we obtain the conclusion that pseudoduality in general sense requires connections ξ and $\tilde{\xi}$ defined respectively on manifolds M and \tilde{M} to be related to each other by the pseudoduality relation (24). Unlike the cases we discussed earlier, curvatures are not constant and the same, but newly defined (modified) curvatures related to modified connections are preserved under the map T . Compared to results found in [5, 6, 9, 10, 11, 16] this does not amount to symmetric

³Cartan structural equation for this modified connection in terms of modified curvature two form is $d\xi_j^i + \xi_k^i \wedge \xi_j^k = (\Omega_\xi)_j^i$. Substituting $\xi_j^i = \omega_j^i - \Gamma_{kj}^i \theta^k$ in this equation one obtains the corresponding Cartan structural equations for ω_j^i and Γ_{kj}^i .

spaces with respect to modified spaces with connections ξ and $\tilde{\xi}$. In the special case that modified connection vanishes (“weak” case above), usual curvature relations are obtained.

To interpret torsions similarly, one needs to take exterior derivative of (22), which leads to

$$T_j^i(D_\xi H_{mn}^j) = (\tilde{D}_{\tilde{\xi}} \tilde{H}_{jk}^i) T_m^j T_n^k \quad (28)$$

where the covariant derivative D_ξ of H_{jk}^i with respect to ξ is defined by

$$D_\xi H_{jk}^i := dH_{jk}^i + H_{jk}^q \xi_q^i - H_{qk}^i \xi_j^q - H_{jq}^i \xi_k^q$$

Taking further exterior derivatives by repeated use of (24) produces infinitely many integrability conditions in terms of covariant derivatives with respect to ξ and $\tilde{\xi}$ for H_{jk}^i and \tilde{H}_{jk}^i

$$\begin{aligned} T_j^i(D_\xi D_\xi H_{mn}^j) &= (\tilde{D}_{\tilde{\xi}} \tilde{D}_{\tilde{\xi}} \tilde{H}_{jk}^i) T_m^j T_n^k \\ T_j^i(D_\xi D_\xi D_\xi H_{mn}^j) &= (\tilde{D}_{\tilde{\xi}} \tilde{D}_{\tilde{\xi}} \tilde{D}_{\tilde{\xi}} \tilde{H}_{jk}^i) T_m^j T_n^k \\ T_j^i(D_\xi \dots D_\xi H_{mn}^j) &= (\tilde{D}_{\tilde{\xi}} \dots \tilde{D}_{\tilde{\xi}} \tilde{H}_{jk}^i) T_m^j T_n^k \end{aligned} \quad (29)$$

Therefore, torsions under pseudoduality are mapped by (22), (28) and (29).

In case of supersymmetric extension of the worldsheet, it is obvious that one can find the same results if the methods and conventions in [9] are followed. This will not be discussed here.

2.2 Case II: Pseudoduality to Symmetric Spaces

Based on above results, an alternative expression which will make use of the equations of motion can be written for pseudoduality as $\tilde{\theta} = {}_{*\Sigma} T\theta$, where ${}_{*\Sigma}$ is the Hodge duality operator, or explicitly

$$\tilde{x}_\tau^i = T_j^i x_\sigma^j \quad (30)$$

$$\tilde{x}_\sigma^i = -T_j^i x_\tau^j \quad (31)$$

Notice that we put a negative sign in the second equation to satisfy the equations of motion. After a little computation it is easy to show that equations (11) and (12) turn to

$$\tilde{x}_{\tau b}^i d\pi^b = [dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\sigma^k + T_j^i x_{\sigma b}^j d\pi^b \quad (32)$$

$$\tilde{x}_{\sigma b}^i d\pi^b = -[dT_k^i - T_j^i \omega_k^j + \tilde{\omega}_j^i T_k^j] x_\tau^k - T_j^i x_{\tau b}^j d\pi^b \quad (33)$$

Wedging first equation by $d\tau$ and second equation by $d\sigma$, and adding together produces two results

$$H = 0 \quad (34)$$

$$dT_k^i + \tilde{\omega}_j^i T_k^j - T_j^i \gamma_k^j = 0 \quad (35)$$

where we used the equations of motion for (M, g, b) and defined a modified connection $\gamma_j^i := \omega_j^i + \Gamma_{kj}^i \theta^k$ on manifold M . Likewise, one wedges the first equation by $d\sigma$ and the second equation by $d\tau$, and subtract from each other to produce the following results

$$\tilde{H} = 0 \quad (36)$$

$$dT_k^i + \tilde{\gamma}_j^i T_k^j - T_j^i \omega_k^j = 0 \quad (37)$$

where we used the equations of motion for sigma model $(\tilde{M}, \tilde{g}, \tilde{b})$ and defined a new modified connection $\tilde{\gamma}_j^i := \tilde{\omega}_j^i + \tilde{\Gamma}_{kj}^i \theta^k$ on manifold \tilde{M} . Thus we understand that this type of pseudoduality kills torsions H and \tilde{H} , and thus gives rise to torsionless manifolds. Actually this explains why Ivanov used torsionless manifolds in his construction [8]. To grasp the remaining equations we subtract (35) from (37) to get

$$\tilde{\Gamma}_j^i T_k^j = -T_j^i \Gamma_k^j \quad (38)$$

This impressive relation is similar to (17) except for the negative sign and actually determines the geometry of the manifolds under pseudoduality. We take the exterior derivative and selectively insert (35) and (37) to get

$$d\tilde{\Omega}_j^i T_k^j = -T_j^i d\Omega_k^j$$

where we used $d\Omega_j^i = d\Gamma_j^i + \Gamma_k^i \wedge \Gamma_j^k$ (and similarly for tilded expression). This gives us a relation between curvatures similar to (20)

$$\tilde{R}_{jkl}^i T_m^j T_n^k T_p^\ell = -T_j^i R_{mnp}^j \quad (39)$$

Intriguing point presents itself when we take one more exterior derivative and again selectively insert (35) and (37) to get $DR_{jkl}^i = \tilde{D}\tilde{R}_{jkl}^i = 0$, where D is the covariant derivative with respect to the connection γ_j^i and \tilde{D} is the covariant derivative with respect to $\tilde{\gamma}_j^i$. Therefore, we obtain that manifolds are symmetric spaces with opposite curvatures. We obtain similar result as in [17] using a different version of pseudoduality equations

In conclusion, we understand that case I pseudoduality arising from the setup $\tilde{\theta} = T\theta$ yields a coordinate diffeomorphism with modified connections, but case II pseudoduality originating from $\tilde{\theta} = {}_{*\Sigma} T\theta$ leads manifolds M and \tilde{M} torsionless and symmetric spaces with opposite curvatures. It is obvious that symmetric space property is a result of hodge duality operator. Unlike the results found in [5, 6, 9, 10], pseudoduality imposes the restriction that manifolds are torsionless.

3 Pseudoduality at Boundaries

We extend our analysis around the boundaries of the region ∂U of the worldsheet Σ . Pseudoduality can be formulated at boundaries by means of the Stokes' theorem, $\int_{\partial U} \theta' = \int_U d\theta'$ where

$d\theta' = \theta$ and θ' is defined at the boundaries. As one might expect pseudoduality equations are reduced to

$$\tilde{x}^i = T_j^i x^j \quad (40)$$

where we use Dirichlet, Neumann or mix mappings $T = \{T_D, T_N, T_M\}$ depending on the type of boundary conditions. If we only have Dirichlet boundary condition (3) then $T = T_D$ is a constant and pseudoduality equations are simply $\tilde{x}^i = x^i$. If we only have the Neumann boundary condition (4), then taking the exterior derivative of (40) with $T = T_N$ gives

$$d\tilde{x}^i = (dT_N)_j^i x^j + (T_N)_j^i dx^j \quad (41)$$

where $\theta' = x$ is taken at boundary ∂U . We define the corresponding connection one form $\omega_j'^i$ and the covariant derivative

$$dx^i + \omega_j'^i x^j = x_a^i d\pi^a$$

Substituting the covariant derivative in (41) together with (40) we obtain

$$\tilde{x}_a^i d\pi^a = [(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j] x^k + (T_N)_j^i x_a^j d\pi^a$$

Subsequently we wedge this expression with $d\sigma$ and $d\tau$ to obtain

$$\begin{aligned} \tilde{x}_\tau^i d\tau \wedge d\sigma &= [(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j] x^k \wedge d\sigma + (T_N)_j^i x_\tau^j d\tau \wedge d\sigma \\ \tilde{x}_\sigma^i d\sigma \wedge d\tau &= [(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j] x^k \wedge d\tau + (T_N)_j^i x_\sigma^j d\sigma \wedge d\tau \end{aligned}$$

Afterwards these two expressions can be inserted in the boundary condition (4) for the $(\tilde{M}, \tilde{g}, \tilde{b})$ to obtain

$$\begin{aligned} -i\tilde{b}_j^i [(dT_N)_k^j + \tilde{\omega}_l'^j (T_N)_k^l - (T_N)_l^j \omega_k'^l] x^k \wedge d\sigma + i\tilde{b}_j^i (T_N)_l^j x_\tau^l d\sigma \wedge d\tau = \\ [(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j] x^k \wedge d\tau + i(T_N)_j^i b_k^j x_\tau^k d\sigma \wedge d\tau \end{aligned} \quad (42)$$

In order to better understand the resultant expression we define the following tensors

$$U_{k\tau}^i d\tau = i\tilde{b}_j^i [(dT_N)_k^j + \tilde{\omega}_l'^j (T_N)_k^l - (T_N)_l^j \omega_k'^l] x^k + i\frac{1}{2}\tilde{b}_j^i (T_N)_k^j x_\tau^k d\tau - i\frac{1}{2}(T_N)_j^i b_k^j x_\tau^k d\tau \quad (43)$$

$$U_{k\sigma}^i d\sigma = [(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j] x^k - i\frac{1}{2}\tilde{b}_j^i (T_N)_k^j x_\tau^k d\sigma + i\frac{1}{2}(T_N)_j^i b_k^j x_\tau^k d\sigma \quad (44)$$

These tensors can be put in (42) to yield $U_{k\tau}^i = U_{k\sigma}^i$. Note that these are the constant tensors. We pull off the minimal case and take them to be zero. It is manifest that splitting $dT_N + \tilde{\omega}'T_N - T_N\omega' = \mathcal{A}'d\sigma + \mathcal{B}'d\tau$ into $d\sigma$ and $d\tau$ directions, and adding and subtracting (43) and (44) one obtains

$$(dT_N)_k^i + \tilde{\omega}_j'^i (T_N)_k^j - (T_N)_j^i \omega_k'^j = 0 \quad (45)$$

$$(T_N)_j^i b_k^j = \tilde{b}_j^i (T_N)_k^j \quad (46)$$

The first result (45) may be proceeded by taking the exterior derivative and considering the integrability conditions to yield the result that curvatures at the boundaries are the same, $R_{jkl}^{'i} = \tilde{R}_{jkl}^{'i}$. One may proceed in a similar way to get an infinite number of relations between covariant derivatives of curvatures. This is not a surprising result since it is an extension of the bulk space results. But interesting result appears in (46) because it describes that pseudoduality at boundaries requires the equality of two form fields b and \tilde{b} while pseudoduality in bulk space demands the equality of torsions (22). Because antisymmetric b and \tilde{b} -fields are locally defined as opposed to the H and \tilde{H} -fields which are globally defined it is understood that pseudoduality at boundaries impose the locality constraint while it is globally defined in the bulk space. This is a natural consequence of the Stokes' theorem which is used to derive the boundary pseudoduality expressions.

One may verify these results in case that (6) and (7) are extended to the boundaries with the restriction (4). Substitute (6) and (7) in $\tilde{x}_\sigma^i - i\tilde{b}_j^i \tilde{x}_\tau^j = 0$ on \tilde{M} and use the same boundary condition $x_\sigma^i - ib_j^i x_\tau^j = 0$ on M to obtain the result (46).

Now we consider the pseudoduality in case that there exist mixed boundary conditions. We introduce the projection operators $\mathcal{P}_{\pm j}^i \equiv \frac{1}{2}(\delta_j^i \pm \mathcal{R}_j^i)$ as in ([13]) where the $(1,1)$ -tensor $\mathcal{R}_j^i(x)$ satisfies

$$\mathcal{R}_k^i \mathcal{R}_j^k = \delta_j^i$$

and leads the metric to be invariant

$$\mathcal{R}_i^j g_{jk} \mathcal{R}_l^k = g_{il}$$

after all it is a symmetric tensor, $\mathcal{R}_{ij} = \mathcal{R}_{ji}$. In other words \mathcal{P}_+ and \mathcal{P}_- project onto the Neumann and Dirichlet directions respectively. Therefore the boundary conditions (3) and (4) can be interpreted as

$$\begin{aligned} \mathcal{P}_{-j}^i \delta x^j &= 0 \\ \mathcal{P}_{+j}^i (x_\sigma^j - ib_k^j x_\tau^k) &= 0 \end{aligned}$$

These equations can also be expressed as follows

$$\delta x^i = \mathcal{P}_{+j}^i \delta x^j \tag{47}$$

$$x_\sigma^i = \mathcal{P}_{-j}^i x_\sigma^j + i\mathcal{P}_{+j}^i b_k^j \mathcal{P}_{+l}^k x_\tau^l \tag{48}$$

where $x_\tau^i = \mathcal{P}_{+j}^i x_\tau^j$ is implemented in (48) if (47) can be put into $\delta x^i = x_\tau^i \delta \tau$ if the time independence is assumed. Furthermore, it is easy to obtain that the projection operator \mathcal{P}_+ is integrable,

$$\mathcal{P}_{+[i}^j \mathcal{P}_{+k]}^l \mathcal{P}_{+j,l}^m = 0$$

This integrability condition requires that the commutator of two infinitesimal displacement in the Neumann direction remains in the Neumann direction, see ([13]). Therefore setting $T = T_M$ and

introducing $\mathcal{S}_{\mp\mp\ell}^i := \mathcal{P}_{\mp j}^i(T_M)_k^j \mathcal{P}_{\mp\ell}^k$ the boundary pseudoduality expression (40) can be written in Dirichlet and Neumann directions respectively

$$\tilde{\mathcal{P}}_{+j}^i \tilde{x}^j = \mathcal{S}_{++\ell}^i x^\ell \quad (49)$$

$$\tilde{\mathcal{P}}_{-j}^i \tilde{x}^j = \mathcal{S}_{--\ell}^i x^\ell \quad (50)$$

with the requirements

$$\mathcal{S}_{+-\ell}^i x^\ell = 0 \quad \text{and} \quad \mathcal{S}_{-+\ell}^i x^\ell = 0$$

Notice that the first expression (49) represents the Dirichlet whereas the second one (50) corresponds to the Neumann boundary conditions. As a result taking δ of the first expression leads to two distinct relations

$$\tilde{\mathcal{P}}_{+j}^i \delta \tilde{x}^j = \mathcal{S}_{++j}^i \delta x^j \quad (51)$$

$$\tilde{\mathcal{P}}_{+j,k}^i (T_M)_\ell^j \mathcal{S}_{++n}^k = \mathcal{S}_{++\ell,m}^i \mathcal{P}_{+n}^m \quad (52)$$

Note that if $T_M = \text{constant}$ is picked, then $\mathcal{S}_{++\ell}^i$ turns to $\mathcal{P}_{+\ell}^i$ and these equations are reduced to the result found above (40). Consequently, pseudoduality causes the dirichlet boundaries to shift by a constant parameter with the condition that the bulk volume remains unchanged⁴.

Now consider the Neumann direction and take the σ -derivative of (50) and use (48) to obtain the following results

$$\tilde{\mathcal{P}}_{-j}^i \tilde{x}_\sigma^j = (T_M)_j^i \mathcal{P}_{-k}^j x_\sigma^k + (T_M)_{j,m}^i \mathcal{P}_{-k}^m x_\sigma^k x^j \quad (53)$$

$$\tilde{\mathfrak{b}}_\ell^i \tilde{x}_\tau^\ell = (T_M)_j^i \mathfrak{b}_\ell^j x_\tau^\ell + (T_M)_{j,m}^i \mathfrak{b}_\ell^m x_\tau^\ell x^j \quad (54)$$

where $\mathfrak{b}_\ell^i := \mathcal{P}_{+j}^i b_k^j \mathcal{P}_{+\ell}^k$ is defined. From the first result (53) one obtains

$$\tilde{\mathcal{P}}_{-j}^i (T_M)_k^j = (T_M)_j^i \mathcal{P}_{-k}^j \quad (55)$$

$$\tilde{\mathcal{P}}_{-j}^i (T_M)_{\ell,k}^j = (T_M)_{\ell,j}^i \mathcal{P}_{-k}^j \quad (56)$$

and the second result (54) yields

$$\tilde{\mathfrak{b}}_j^i (T_M)_k^j = (T_M)_j^i \mathfrak{b}_k^j \quad (57)$$

$$\tilde{\mathfrak{b}}_j^i (T_M)_{k,\ell}^j = (T_M)_{k,j}^i \mathfrak{b}_\ell^j \quad (58)$$

These are all the relations that determine the boundary pseudoduality equations in case of mixed boundary conditions. For the trivial case where T_M is a constant, pseudoduality equations in Dirichlet and Neumann directions simply become $\tilde{\mathcal{P}}_{\pm j}^i \tilde{x}^j = \mathcal{P}_{\pm j}^i x^j$ with the following conditions

$$\begin{aligned} \tilde{\mathcal{P}}_{+j}^i \delta \tilde{x}^j &= \mathcal{P}_{+j}^i \delta x^j & \tilde{\mathcal{P}}_{+j,k}^i &= \mathcal{P}_{+j,k}^i \\ \tilde{\mathcal{P}}_{-j}^i &= \mathcal{P}_{-j}^i & \tilde{\mathfrak{b}}_j^i &= \mathfrak{b}_j^i \end{aligned}$$

⁴Notice that (52) gives $\tilde{\mathcal{P}}_{+j,k}^i = \mathcal{P}_{+j,k}^i$.

Notice that Dirichlet projection operator is preserved while Neumann projection operator satisfies the conditions in the first line. Locality constraint is obvious and given by the equality of “projected” antisymmetric \mathfrak{b} -fields.

4 Pseudoduality in WZW Models

Analysis we established in above sections can be carried out for sigma models based on group manifolds. We emphasize that group manifolds are mappings from Euclidean worldsheets. This fixes the problems we encountered in supersymmetric cases in [9]. Let us consider a strict WZW sigma model [14] based on a compact Lie group G of dimension n . Lagrangian of this model is given by

$$\mathcal{L} = \frac{1}{2} \text{Tr}(\partial_\mu g^{-1} \partial^\mu g) + \Gamma \quad (59)$$

where Γ represents the WZ term, and the field $g(\tau, \sigma)$ defined on the Euclidean space (possibly with $-\infty < \sigma \leq 0$) [15] takes values in a compact classical Lie Group G and is given by the map $g : \Sigma \rightarrow G$. There is a global continuous $G \times G$ symmetry $g \rightarrow UgV^{-1}$, $U, V \in G$, which gives \mathfrak{g} -valued⁵ conserved currents with zero curvature

$$j_\mu^R = g^{-1} \partial_\mu g, \quad j_\mu^L = -\partial_\mu g g^{-1} \quad (60)$$

The equations of motion in bulk space are $\partial^\mu j_\mu^R = \partial^\mu j_\mu^L = 0$. The boundary equation of motion at $\sigma = 0$ is

$$\text{Tr}(g^{-1} \partial_\sigma g g^{-1} \delta g) = 0 \quad (61)$$

where $g^{-1} \delta g \in \mathfrak{g}$. Obviously one obtains the Dirichlet and Neumann boundary conditions respectively

$$\delta g = 0, \quad \partial_\sigma g = 0 \quad (62)$$

Notice that boundary equation of motion is equivalent to $\text{Tr}(j_\tau, j_\sigma) = 0$. We know that both currents generate the orthonormal coframes $\{j\}$ on the pullback bundle $g^*(TG)$ such that they satisfy the Maurer-Cartan equation

$$dj^i + \frac{1}{2} f_{k\ell}^i j^k \wedge j^\ell = 0$$

where $\theta^i = j^i$, $j^i = j_a^i d\pi^a = (g^{-1} \partial_a g)^i d\pi^a$ and $\omega_\ell^i = \frac{1}{2} f_{k\ell}^i j^k$ is the antisymmetric riemannian connection. These coframes together with the corresponding riemannian connection satisfy the Cartan structural equations (9)-(10). The bulk space pseudoduality equations in the first case

⁵ \mathfrak{g} is the Lie algebra of G with a negative-definite invariant inner product $\langle \cdot, \cdot \rangle$.

are given by $\tilde{j}^R = Tj^R$ and expressed in the following forms

$$\tilde{j}_\tau^R = Tj_\tau^R \quad (63)$$

$$\tilde{j}_\sigma^R = Tj_\sigma^R \quad (64)$$

We inquire the integrability conditions by taking ∂_τ derivative of (63), ∂_σ derivative of (64) and adding them together to obtain

$$T^{-1}(\partial_\mu T)j_\mu^R = T^{-1}(\partial_\tau T)j_\tau^R + T^{-1}(\partial_\sigma T)j_\sigma^R = 0 \quad (65)$$

Obviously T depends on currents j_τ^R and j_σ^R nonlinearly and solution requires using the identity [16]

$$T^{-1}(\partial_\mu T) = \frac{1 - e^{-adX}}{adX} \partial_\mu X = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} [X, \dots [X, \partial_\mu X]]$$

where we introduced an exponential solution $T = e^X$ ⁶, and $adX : \mathfrak{g} \rightarrow \mathfrak{g}$, the adjoint representation of X , and $adX(Y) = [X, Y]$, $\forall Y \in \mathfrak{g}$. We let $X \rightarrow \epsilon X$ for small parameter ϵ and look for a perturbation solution to get

$$(\partial_\tau X)j_\tau^R + (\partial_\sigma X)j_\sigma^R = 0$$

in the first order of ϵ . Trivial solution is that X is a constant so that T may be chosen to be identity. Therefore, pseudoduality maps the group manifold G to itself. The general solution requires tedious analysis, and we just consider a restricted solution for simplicity. Assume that both terms of partial differential equation is independent of each other so that solution for the lie algebra valued X can be written as

$$X = \left\{ \int j_\tau^{-1} d\tau \cup \int j_\sigma^{-1} d\sigma \right\} \quad (66)$$

where we dropped the upper label R for convenience. Therefore, infinite number of pseudodual currents can be written in terms of nonlocal currents using the pseudoduality expressions (63) and (64)

$$\tilde{j}_\mu = \sum_{n=1}^{\infty} \epsilon^n \tilde{j}_\mu^{(n)} \quad (67)$$

Another intriguing result of special importance is the commutation relations between currents living on pseudodual manifold and is obtained by taking ∂_σ of (63), ∂_τ of (64) and subtracting from each other

$$[\tilde{j}_\tau, \tilde{j}_\sigma]_{\tilde{G}} = T[j_\tau, j_\sigma]_G + (\partial_\sigma T)j_\tau - (\partial_\tau T)j_\sigma \quad (68)$$

where $[\cdot, \cdot]_G$ and $[\cdot, \cdot]_{\tilde{G}}$ are bracket relations in G and \tilde{G} respectively. Once we find the solution for T using (66), we insert in (68) and come up with the bracket relation on the pseudodual manifold \tilde{G} .

⁶ $X \in so(n)$, the Lie algebra of $SO(n)$.

In the second case pseudoduality equations are expressed by $\tilde{j} =_{*\Sigma} Tj$ and are written explicitly as

$$\tilde{j}_\tau = Tj_\sigma \quad (69)$$

$$\tilde{j}_\sigma = -Tj_\tau \quad (70)$$

where j stands for both j^R and j^L . Taking ∂_τ of (69), ∂_σ of (70) and adding together yields the following commutation relation which gives rise to a solution for T

$$[j_\sigma, j_\tau]_G = (T^{-1}\partial_\sigma T)j_\tau - (T^{-1}\partial_\tau T)j_\sigma \quad (71)$$

This is just a special case of (68) when the commutation relation of the pseudodual currents vanish, i.e. $[\tilde{j}_\tau, \tilde{j}_\sigma]_{\tilde{G}} = 0$. Likewise one takes ∂_σ of (69), ∂_τ of (70) and subtract to get a commutation relation of pseudodual currents

$$[\tilde{j}_\tau, \tilde{j}_\sigma]_{\tilde{G}} = (\partial^\mu T)j_\mu = (\partial_\sigma T)j_\sigma + (\partial_\tau T)j_\tau \quad (72)$$

Hence, using above expansion for $T^{-1}\partial_\mu T$, one finds out a solution of T in (71) in terms of currents j_τ, j_σ and commutation relation $[j_\sigma, j_\tau]_G$ and puts this solution in (72) to obtain a commutation relation of the pseudodual currents as an infinite number of series in terms of currents on the generic manifold M . In fact, case II pseudoduality generates an infinite number of commutation relations while case I pseudoduality just yields an infinite number of currents in terms of currents on manifold M . To find certain expressions for commutation relations, a specific solution should be chosen.

Pseudoduality can be extended to boundaries using Stokes' theorem as above section to obtain the boundary pseudoduality expression in a simple form

$$\tilde{g} = Tg \quad (73)$$

It is obvious that if there is only Dirichlet boundary condition, then T is trivial, and identity. If we only have Neumann boundary condition, then taking ∂_σ yields that T only depends on τ . It can be any τ -dependent function so that pseudoduality conditions are satisfied. Therefore, one obtains the currents at boundaries

$$\tilde{j}_\sigma = j_\sigma, \quad \tilde{j}_\tau = j_\tau + g^{-1}(T^{-1}\partial_\tau T)g \quad \text{at boundary (at } \sigma = 0)$$

In the presence of mixed boundary conditions one needs to perform analysis on the symmetric spaces. This can be accomplished using above reasoning and results in [16].

5 Concluding Remarks

We performed all possible pseudoduality transformations between different sigma models with Euclidean signatures and extended our analysis to boundaries. We have seen that there could be two different types of pseudodualities, each of which produced intriguing results reflecting their own peculiarities. In case of pseudoduality producing coordinate diffeomorphism, Integrability conditions led us to define modified connections ξ and $\tilde{\xi}$ on manifolds M and \tilde{M} respectively. These connections provided to find out a general pseudoduality condition (24), which yielded a relation between curvatures with respect to the modified connections. We also obtained infinitely many torsion relations and their covariant derivatives as given in (22), (28) and (29). In the special case that these modified connections vanish, we found out that torsions and curvature tensors are preserved under pseudoduality and produce the coordinate diffeomorphisms. Case II pseudoduality more likely concerns the geometry of the manifolds and resulted in a conclusion about the geometry of manifolds M and \tilde{M} to be dual symmetric spaces with respect to modified connections ξ and $\tilde{\xi}$ respectively. This results in a conclusion that pseudoduality imposes the manifolds be symmetric spaces with opposite curvatures. This is the generic feature of case II pseudoduality, not just special to sigma models due to (51) and (53). This type of pseudoduality does not allow manifolds M and \tilde{M} torsionful.

We have demonstrated that boundary pseudoduality gives the locality constraint and preserves the antisymmetric two-form field. Boundary pseudoduality analysis leads to a convenient framework for the pseudoduality of D-Branes.

Sigma models based on group manifolds yield an infinite number of nonlocal conserved currents under case I pseudoduality (67). We also obtained the commutation relations between manifolds G and \tilde{G} (68). Case II pseudoduality leads to appropriate commutation relations of currents on both manifolds M and \tilde{M} . Commutation relations on \tilde{M} are expressed by infinite number of terms as functions of currents and their commutation relations on manifold M . Boundary conditions are used to find currents at boundaries.

In general, since pseudoduality is performed on the worldsheets integrability conditions are determined by the metric of worldsheet. It turns out that Euclidean metric yields well-defined results compared to Lorentzian metric when worldsheet is supersymmetrized. It is also intriguing to construct pseudoduality on worldsheets with a general metric. We plan to explore this case in a more general context.

A Appendix

Lorentzian action corresponding to (1) is given by

$$S = \int d\tau d\sigma \left(\frac{1}{2} g_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} - \frac{1}{2} g_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \sigma} \right) + b_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma} \quad (74)$$

where the functions $x^i(\sigma)$ giving the dependence of the real coordinates x^i of M on the real coordinates σ^μ of Σ . The worldsheet Σ is endowed with the Lorentzian metric $h_{\mu\nu}$. Notice that this Lorentzian action is real. The bulks space equations of motion following from this action will be

$$x_{\sigma\sigma}^i - x_{\tau\tau}^i = -\Gamma_{jk}^i (x_\tau^j x_\tau^k - x_\sigma^j x_\sigma^k) - H_{jk}^i x_\sigma^j x_\tau^k \quad (75)$$

with the corresponding Dirichlet and Neumann boundary conditions respectively

$$\delta x^i = 0 \quad (76)$$

$$x_\sigma^i - b_j^i x_\tau^j = 0 \quad (77)$$

Pseudoduality equations are stated with $\tilde{\theta} = {}_{*\Sigma} T\theta$, where ${}_{*\Sigma}$ denotes the Hodge duality operator, and given by the following pairs of equations

$$\tilde{x}_\tau = T x_\sigma$$

$$\tilde{x}_\sigma = T x_\tau$$

Therefore, particle-like solutions (σ -independent) on M get mapped into static soliton-like solutions on \tilde{M} and vice-versa. Integrability conditions for these equations yield that torsions of both manifolds M and \tilde{M} vanish, and when extended to supersymmetry, pseudoduality transformation is not invertible and not well-defined globally [9, 10, 11, 17].

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